6.S965
Digital Systems Laboratory II

Lecture 9:

IQ and related topics

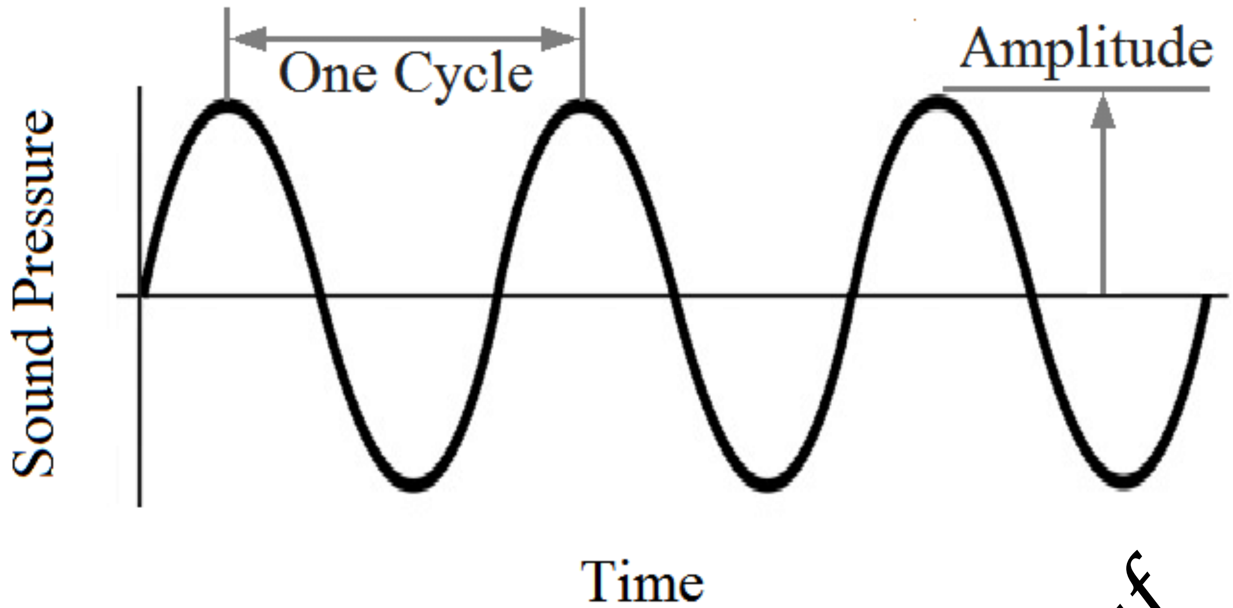
Administrative

- Week 4 due Friday
- Week 5 will come out on this upcoming Friday:
  - Two parts:
    - Build a AXIS-Magnitude/Angle Finder (16-stage CORDIC)
    - Test it
  - AND:
    - Add a DMA readout to your lab 3 system to grab lines of video (don’t need CORDIC for this)

Motivation

- One of the reasons I wanted to look at CORDIC stuff was it would let us think more about doing trig functions and other operations
- Forms an important part of a lot of how FPGAs are used, particularly in signal processing applications.
- A lot of signals come in and you need to do very quick math to extract/refine the information from them.

Sine wave looks like this



$$v(t) = A\sin(\omega t + \phi)$$

$$\omega = 2\pi f$$

If you needed to convey information on this wave what could you do?

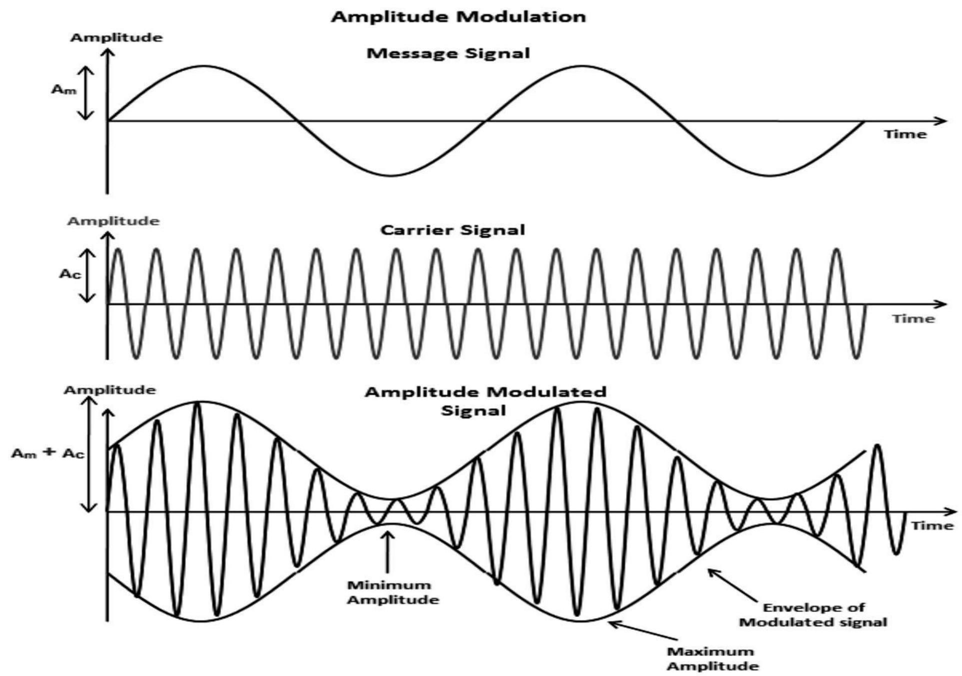
$$v(t) = A\cos(\omega t + \phi)$$

- You could:
  - Vary the amplitude (Amplitude modulation)

$$v(t) = \color{red}{A(t)}\cos(\omega t + \phi)$$

Amplitude Modulation

- Keep frequency the same and then modulate the amplitude of your carrier wave…
- Usually something as simple as a low-pass filter and some non-linearity can get the info out



https://byjus.com/jee/amplitude-modulation/

Some Math… $v(t) = A\cos(\omega t + \phi)$

- If we are varying our amplitude over time to convey information, then we have this:

$$v(t) = A(t)\cos(\omega t + \phi)$$

- Because $A(t)$ is just a time-varying signal, and all time-varying signals can be represented with sum of weighted sinusoids (thanks Fourier), we can get a lot of insight by just thinking about $A(t) = A \cdot \cos(\omega_m t)$

*for simplicity, we'll assume there's no offset phase in the modulating signal or carrier

A Simple AM signal therefore looks like:

$$v(t) = (A \cdot \cos(\omega_m t)) \cdot \cos(\omega_c t)$$

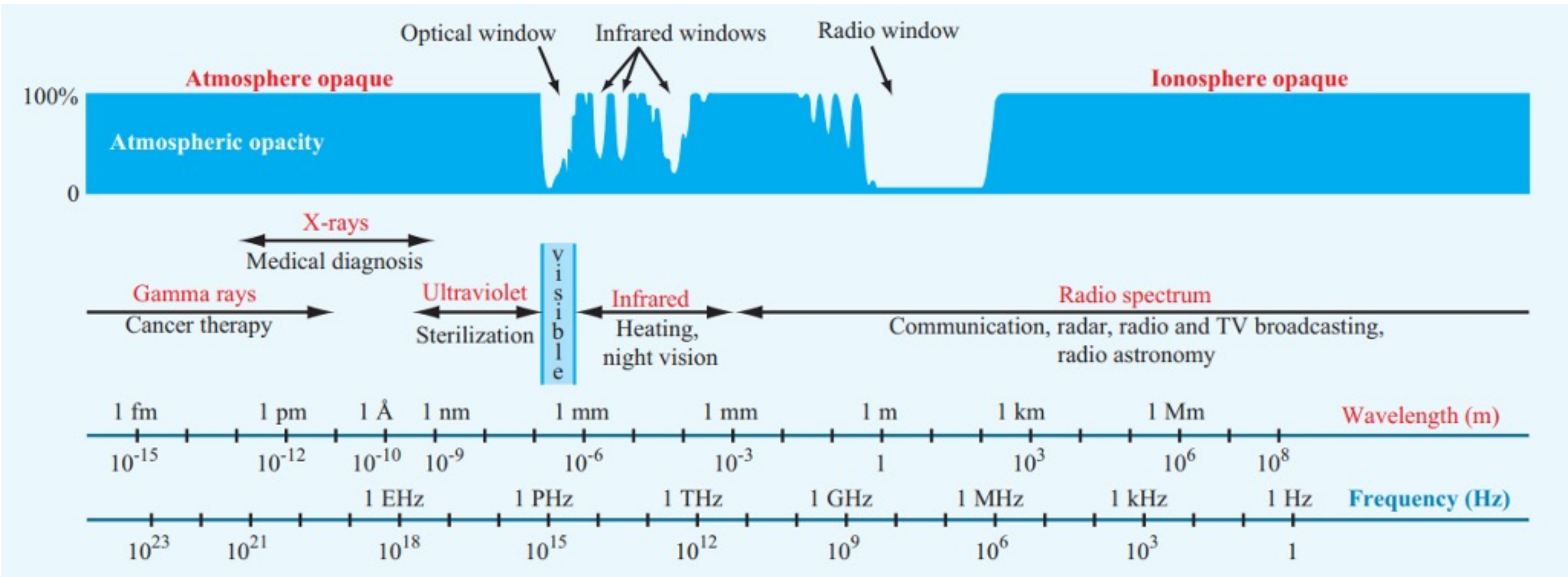
- ω_m is a frequency of our message (voice, data, whatever)
- ω_c is the frequency of our CARRIER WAVE.

- What makes a good carrier wave?

https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Product-to-sum_and_sum-to-product_identities

Carrier Waves

- Lots of reasons

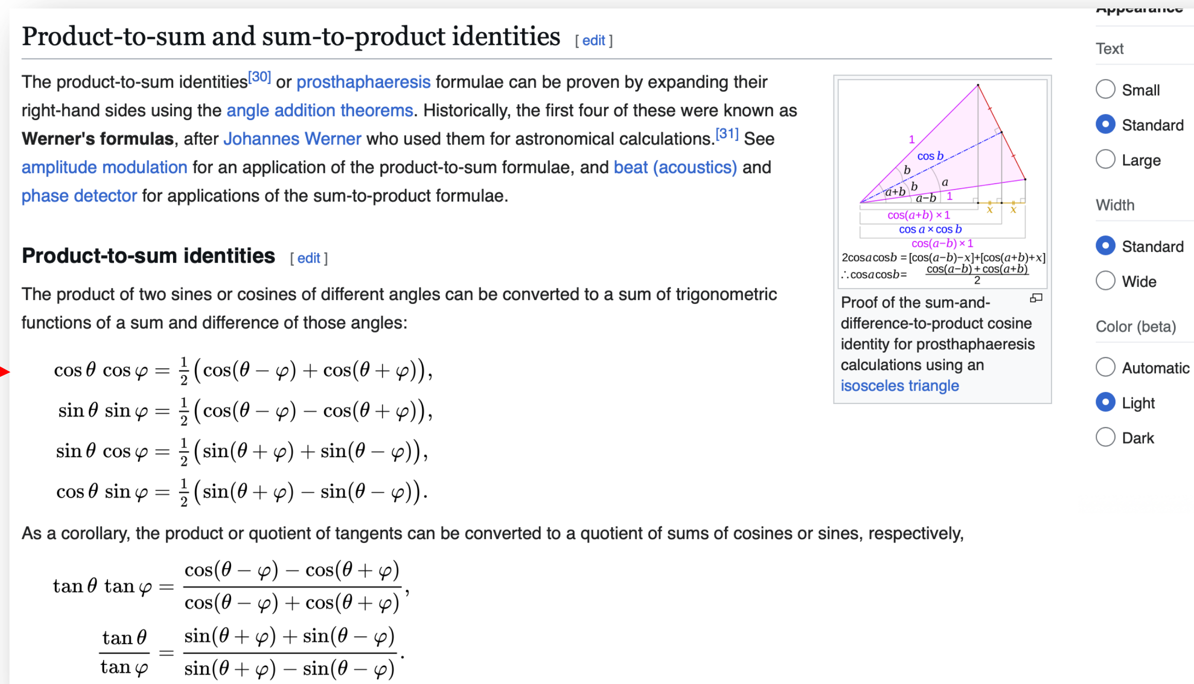


https://www.nasa.gov/smallsat-institute/sst-soa/soa-communications/

An AM signal therefore looks like:

$$v(t) = A \cdot \cos(\omega_m t) \cdot \cos(\omega_c t)$$

- Using some of these nasties…:



Product-to-sum and sum-to-product identities [edit]

The product-to-sum identities[30] or prosthaphaeresis formulae can be proven by expanding their right-hand sides using the angle addition theorems. Historically, the first four of these were known as **Werner's formulas**, after Johannes Werner who used them for astronomical calculations.[31] See amplitude modulation for an application of the product-to-sum formulae, and beat (acoustics) and phase detector for applications of the sum-to-product formulae.

Proof of the sum-and-difference-to-product cosine identity for prosthaphaeresis calculations using an isosceles triangle

Product-to-sum identities [edit]

The product of two sines or cosines of different angles can be converted to a sum of trigonometric functions of a sum and difference of those angles:

$$\cos\theta \cos\varphi = \tfrac{1}{2}\big(\cos(\theta-\varphi) + \cos(\theta+\varphi)\big),$$
$$\sin\theta \sin\varphi = \tfrac{1}{2}\big(\cos(\theta-\varphi) - \cos(\theta+\varphi)\big),$$
$$\sin\theta \cos\varphi = \tfrac{1}{2}\big(\sin(\theta+\varphi) + \sin(\theta-\varphi)\big),$$
$$\cos\theta \sin\varphi = \tfrac{1}{2}\big(\sin(\theta+\varphi) - \sin(\theta-\varphi)\big).$$

As a corollary, the product or quotient of tangents can be converted to a quotient of sums of cosines or sines, respectively,

$$\tan\theta \tan\varphi = \frac{\cos(\theta-\varphi) - \cos(\theta+\varphi)}{\cos(\theta-\varphi) + \cos(\theta+\varphi)},$$
$$\frac{\tan\theta}{\tan\varphi} = \frac{\sin(\theta+\varphi) + \sin(\theta-\varphi)}{\sin(\theta+\varphi) - \sin(\theta-\varphi)}.$$

Appearance

Text

Small

Standard

Large

Width

Standard

Wide

Color (beta)

Automatic

Light

Dark

https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Product-to-sum_and_sum-to-product_identities

$$\cos\theta\ \cos\varphi = \tfrac{1}{2}\big(\cos(\theta-\varphi)+\cos(\theta+\varphi)\big),$$

So then…

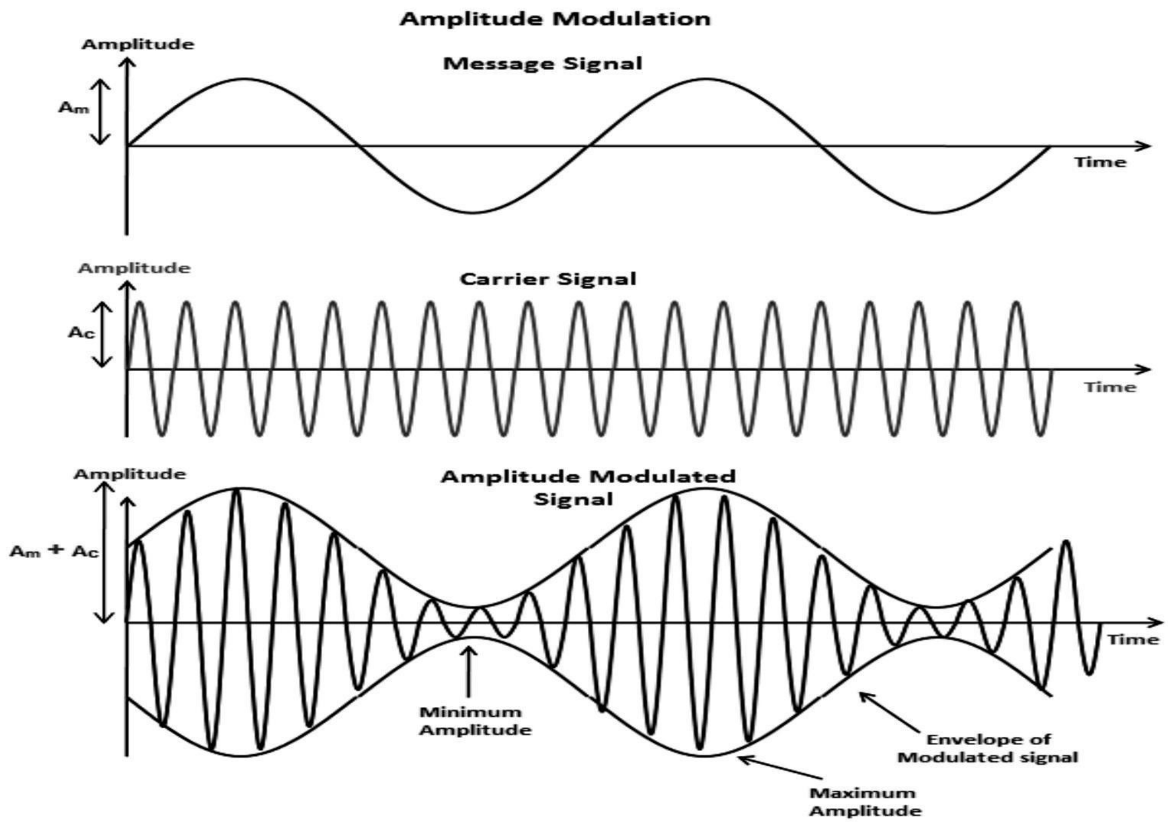
$$v(t) = A \cdot \cos(\omega_m t) \cdot \cos(\omega_c t)$$

- Becomes…

$$v(t) = \frac{A}{2} \cdot \cos\big((\omega_c + \omega_m)t\big) + \frac{A}{2} \cdot \cos\big((\omega_c - \omega_m)t\big)$$

Amplitude Modulation

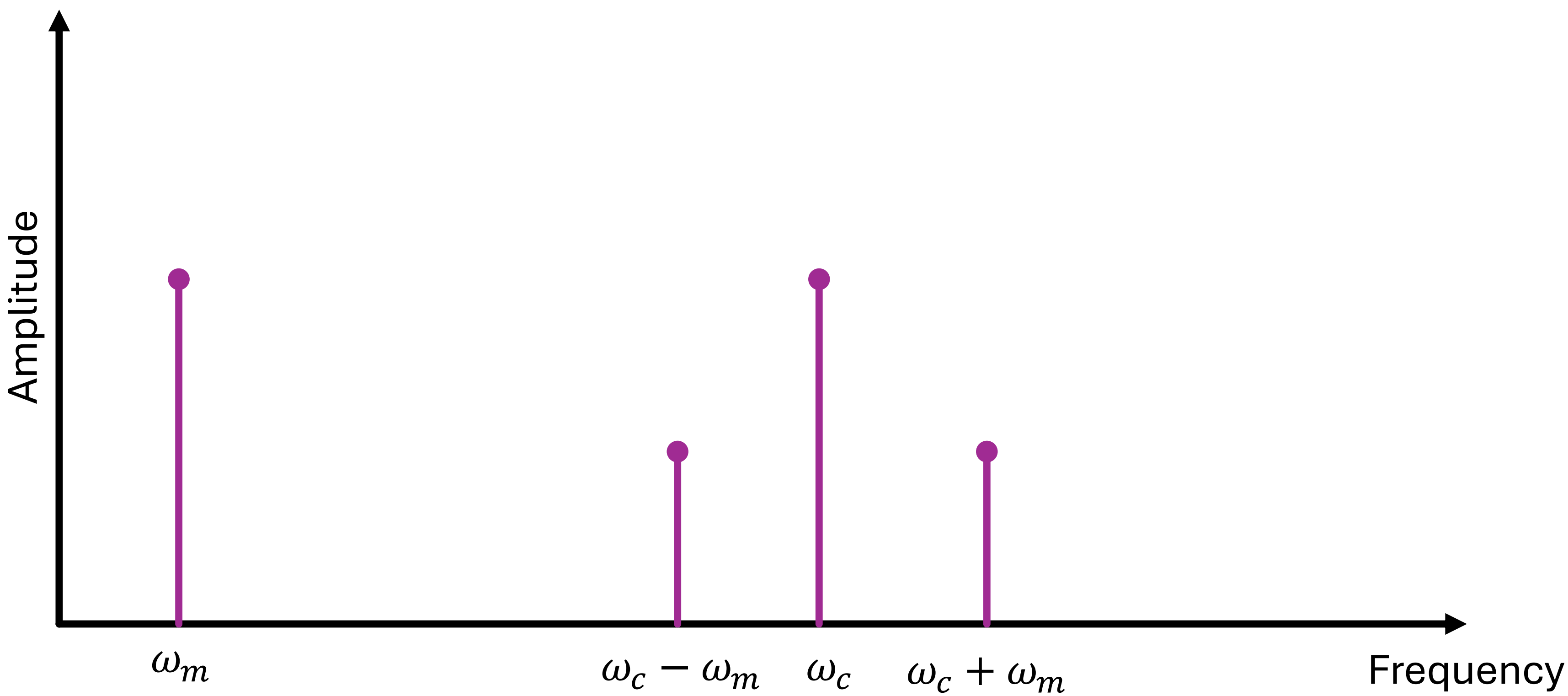
$$v(t) = \frac{A}{2} \cdot \cos\big((\omega_c + \omega_m)t\big) + \frac{A}{2} \cdot \cos\big((\omega_c - \omega_m)t\big)$$



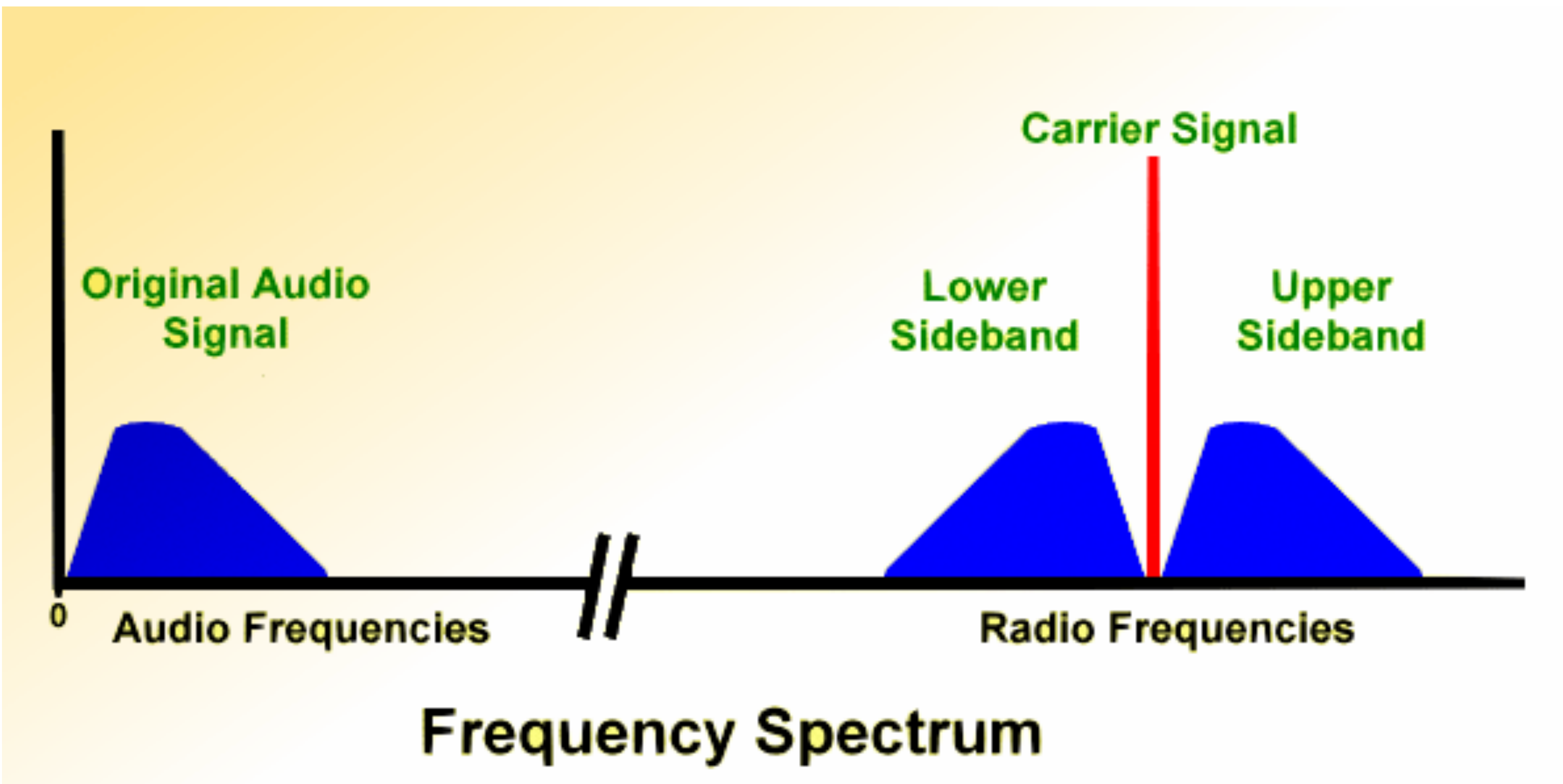
https://byjus.com/jee/amplitude-modulation/

Interesting observation

- To AM one frequency, you end up generating *two frequencies…good or bad?*

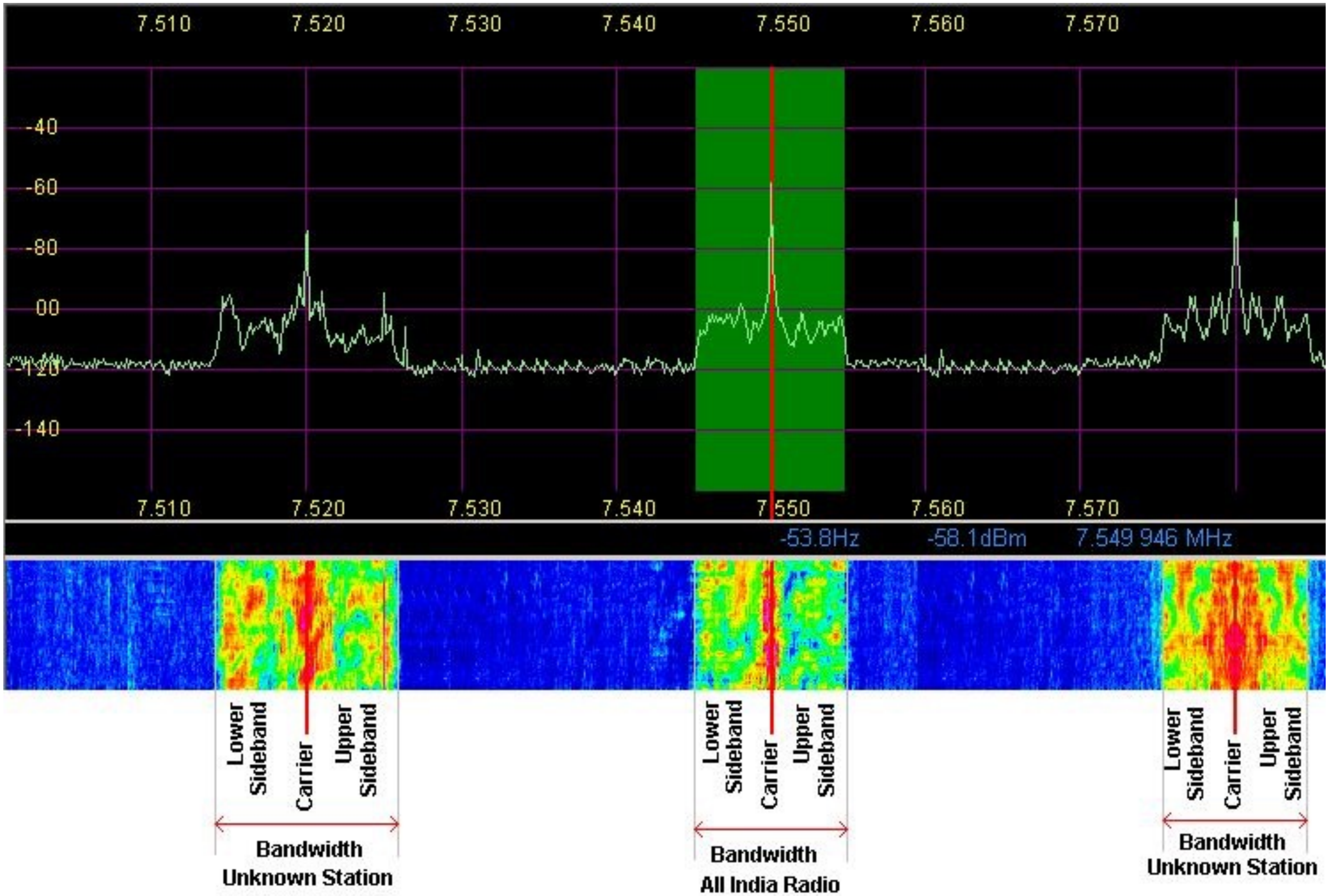


So that means…transmitting a range of frequencies in a message with AM will audio will take up twice the bandwidth



https://vu2nsb.com/wp-content/uploads/2020/03/sideband.png

This is indeed how it looks



https://reviseomatic.org/help/2-radio/Radio%20Frequency%20Bands.php

Often times there won't be full modulation

- Doing this:

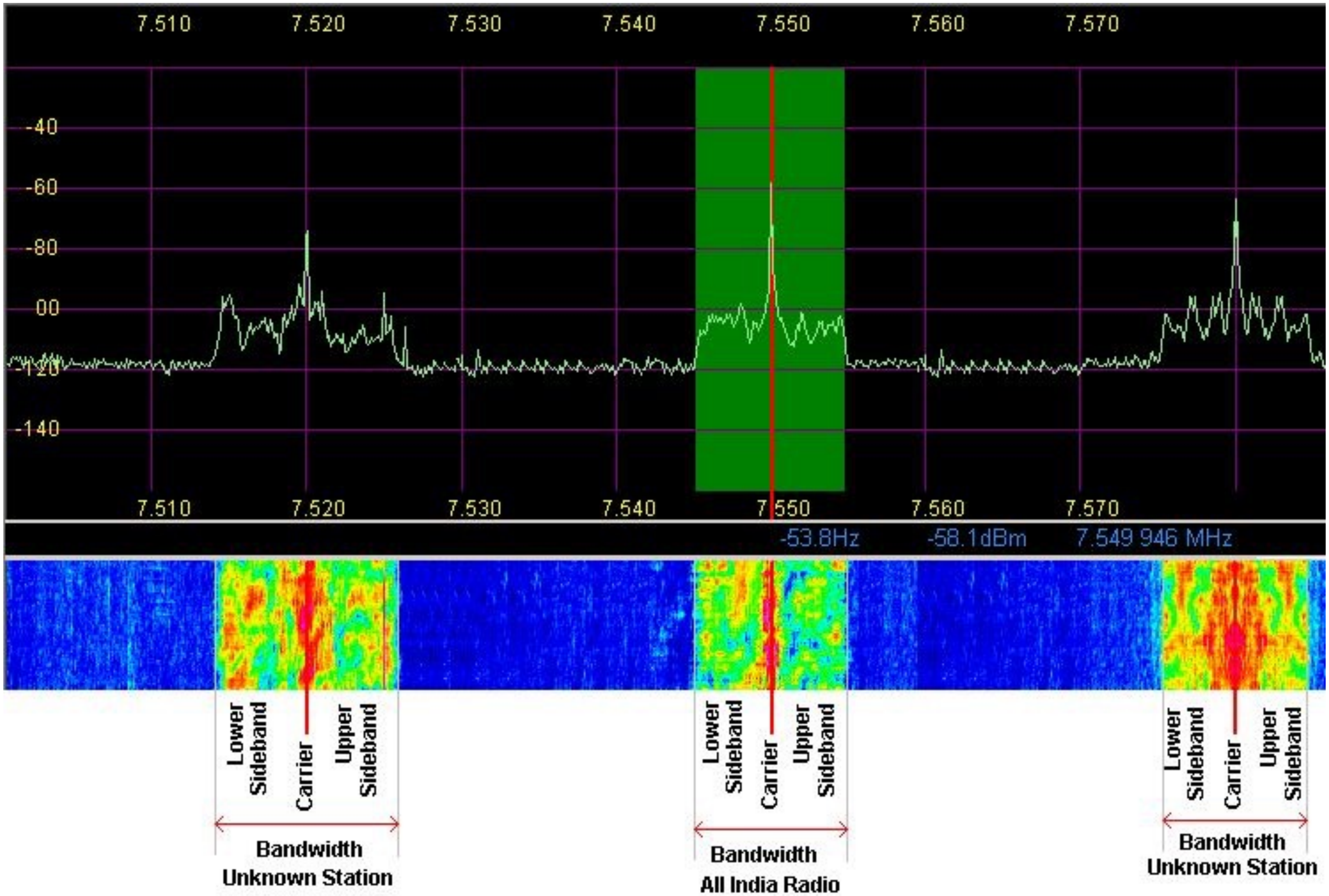
$$v(t) = A(t)\cos(\omega_c t)$$

- Means 100% of the carrier wave gets modulated.
- More often you'll see:

$$v(t) = (1 + A(t))\cos(\omega_c t)$$

- So some of the purified carrier just stays at the carrier frequency

This is indeed how it looks



https://reviseomatic.org/help/2-radio/Radio%20Frequency%20Bands.php

To harvest the information from that modulated wave…

- Capture the wave from its medium:

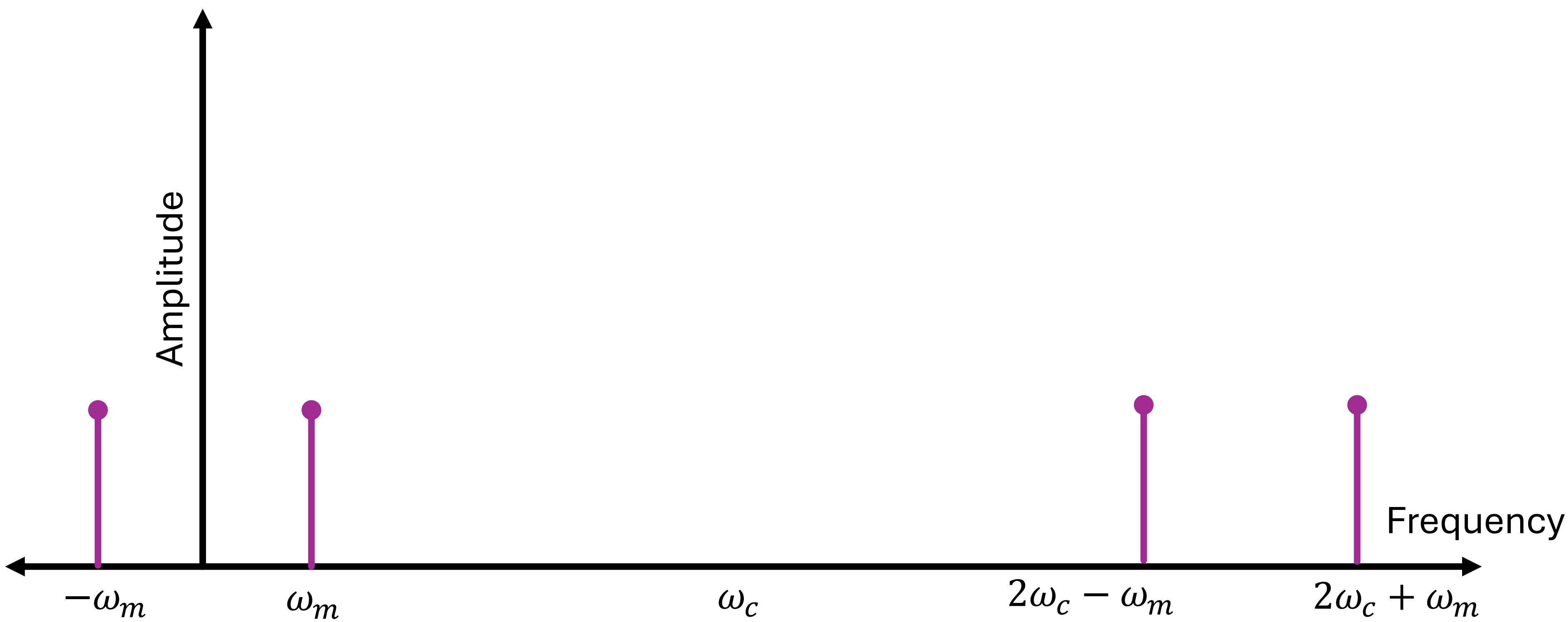
$$v(t) = \frac{A}{2} \cdot \cos\big((\omega_c + \omega_m)t\big) + \frac{A}{2} \cdot \cos\big((\omega_c - \omega_m)t\big)$$

- Multiply it again by the targeted carrier wave $\cos(\omega_c t)$ (just like before to get four terms…)

$$v(t) = \frac{A}{4} \cdot \cos\big((\omega_c + \omega_m + \omega_c)t\big) + \frac{A}{4} \cdot \cos\big((\omega_c + \omega_m - \omega_c)t\big)$$
$$+ \frac{A}{4} \cdot \cos\big((\omega_c - \omega_m + \omega_c)t\big) + \frac{A}{4} \cdot \cos\big((\omega_c - \omega_m - \omega_c)t\big)$$

And then… You get four different terms…

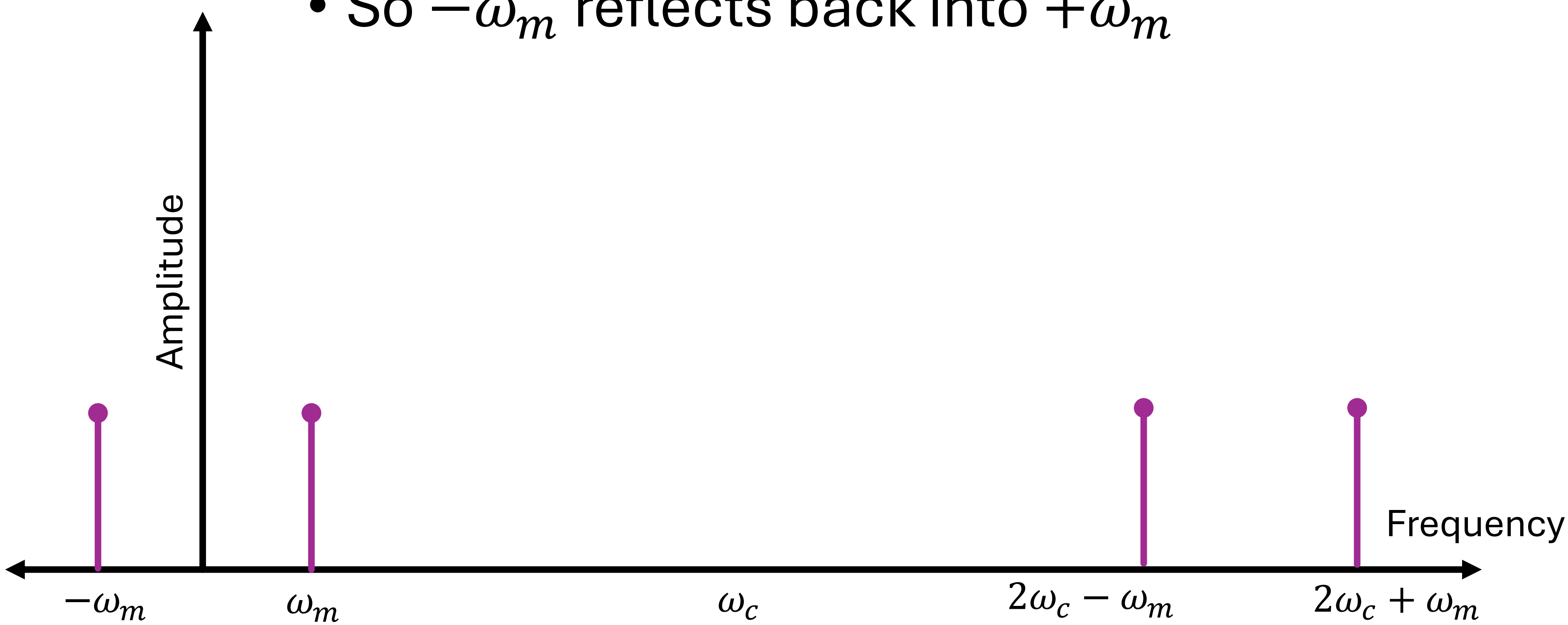
$$v(t) = \frac{A}{4} \cdot \cos\big((\omega_c + \omega_m + \omega_c)t\big) + \frac{A}{4} \cdot \cos\big((\omega_c + \omega_m - \omega_c)t\big)$$
$$+ \frac{A}{4} \cdot \cos\big((\omega_c - \omega_m + \omega_c)t\big) + \frac{A}{4} \cdot \cos\big((\omega_c - \omega_m - \omega_c)t\big)$$



Cos(-x) = cos(x)

$$v(t) = \frac{A}{4} \cdot \cos\big((\omega_c + \omega_m + \omega_c)t\big) + \frac{A}{4} \cdot \cos\big((\omega_c + \omega_m - \omega_c)t\big)$$
$$+ \frac{A}{4} \cdot \cos\big((\omega_c - \omega_m + \omega_c)t\big) + \frac{A}{4} \cdot \cos\big((\omega_c - \omega_m - \omega_c)t\big)$$

- So $-\omega_m$ reflects back into $+\omega_m$



LPF away that extra trash up high…

$$v(t) = \frac{A}{4} \cdot \cos\big((\omega_c + \omega_m + \omega_c)t\big) + \frac{A}{4} \cdot \cos\big((\omega_c + \omega_m - \omega_c)t\big)$$
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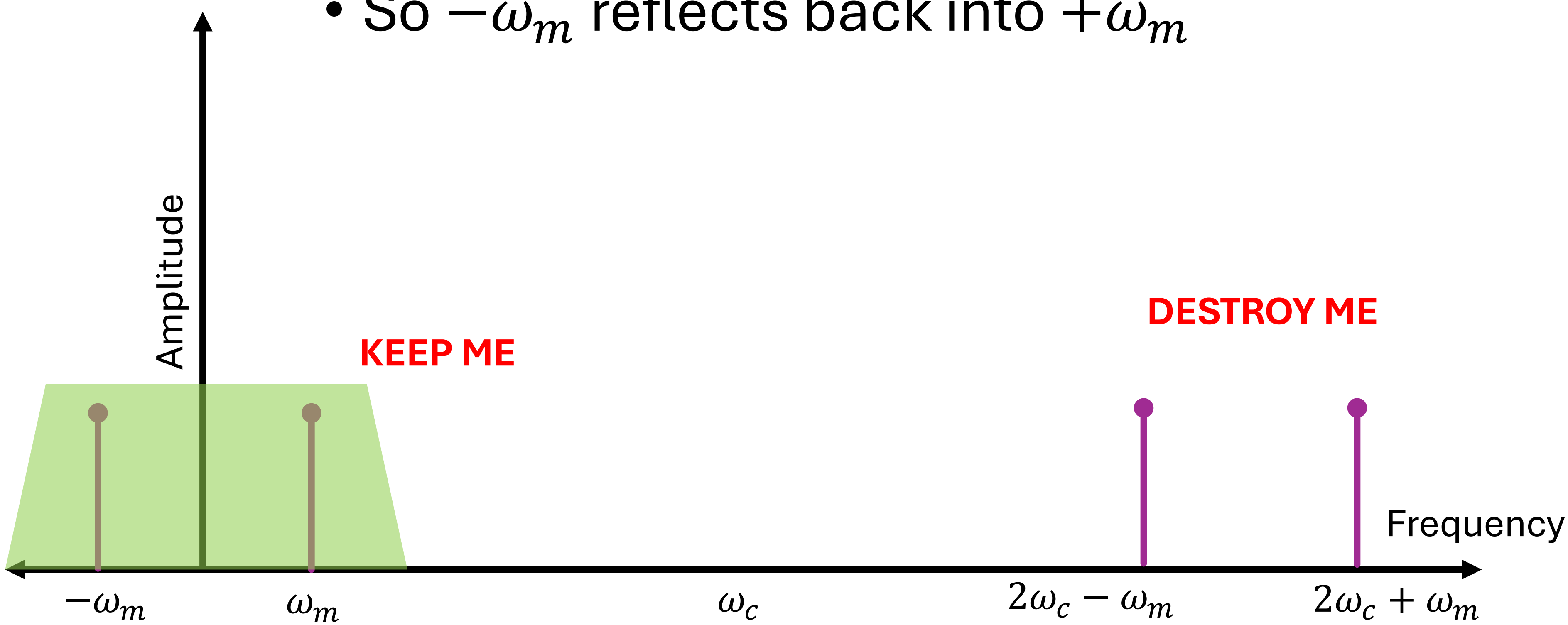


Couple Things to Think About

- AM transmits redundant information
  - You can in fact suppress one of the sidebands (called single-side-band transmission), using only the bandwidth of your message, but recovery can be harder since you have to perfectly line up your local oscillator.
  - The presence of some carrier is useful for signal recovery

Couple Things to Think About

- What's up with that negative frequency? Does that mean anything or are we totally ok ignoring it and assuming the world will work out ok.

- Yes/No

Is AM used Much?

$$v(t) = A(t)\cos(\omega t + \phi)$$

- It has lots of downsides…the big one is that you’re using the one dimension of a wave that is highly highly a function of things like:
  - Distance traveled
  - General noise
- Also “AM” was historically used with an analog systems which were just inherently more prone to noise
- But in a more global sense, it is still used a lot, just not exactly how we’re presenting it here.

If you needed to convey information on this wave what could you do?

$$v(t) = A\cos(\omega t + \phi)$$

- You could:
  - Vary the frequency (Frequency modulation)

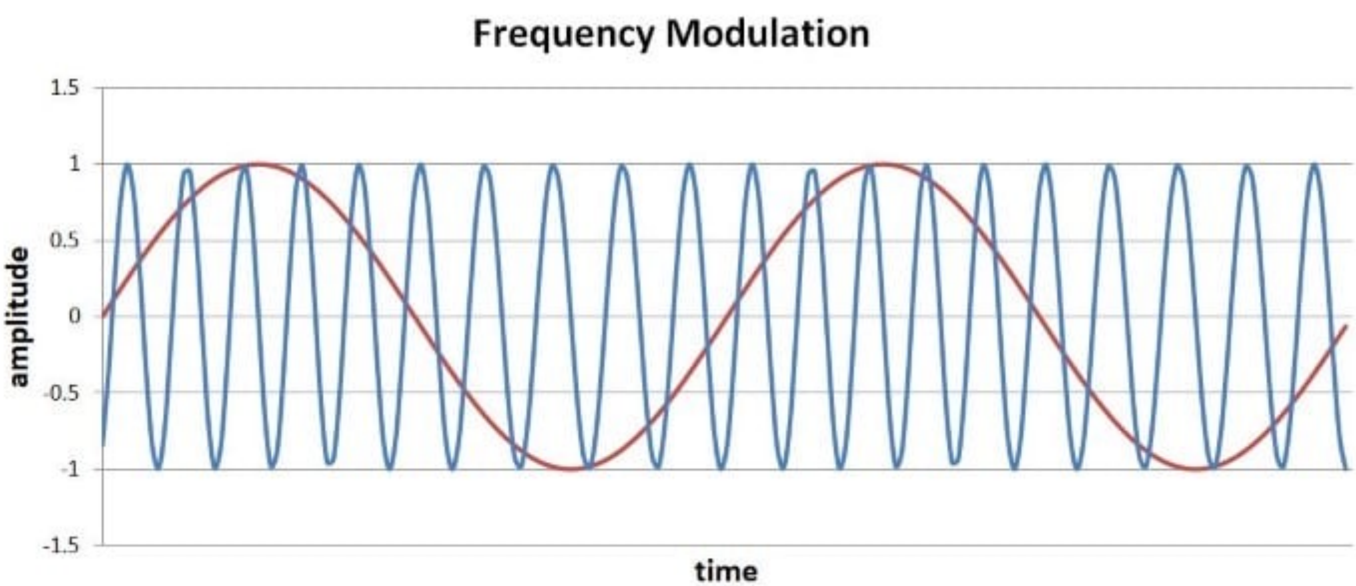
$$v(t) = A\cos(\omega(t)t + \phi)$$

$$v(t) = A\cos\big((\omega_c + \cos(\omega_m t))t\big)$$

**again, ignore phase for this one for simplicity.*

Frequency Modulation

- You keep a constant amplitude and then vary your frequency around a constant center frequency
- Circuits can extract those deviations to get the info



https://www.allaboutcircuits.com/textbook/radio-frequency-analysis-design/radio-frequency-modulation/frequency-modulation-theory-time-domain-frequency-domain/

FM Modulation Looks Like…

$$v(t) = A\cos\big((\omega_c + \cos(\omega_m t))t\big)$$

Angle sum and difference identities [edit]

See also: Proofs of trigonometric identities § Angle sum identities, and Small-angle ap

These are also known as the *angle addition and subtraction theorems* (or *formulae*).

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \mathrm{c}$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \mathrm{c}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \mathrm{s}$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \mathrm{s}$$

Composition of trigonometric functions [edit]

These identities involve a trigonometric function of a trigonometric function:[56]

$$\cos(t\sin x) = J_0(t) + 2\sum_{k=1}^{\infty} J_{2k}(t)\cos(2kx)$$

$$\sin(t\sin x) = 2\sum_{k=0}^{\infty} J_{2k+1}(t)\sin\big((2k+1)x\big)$$

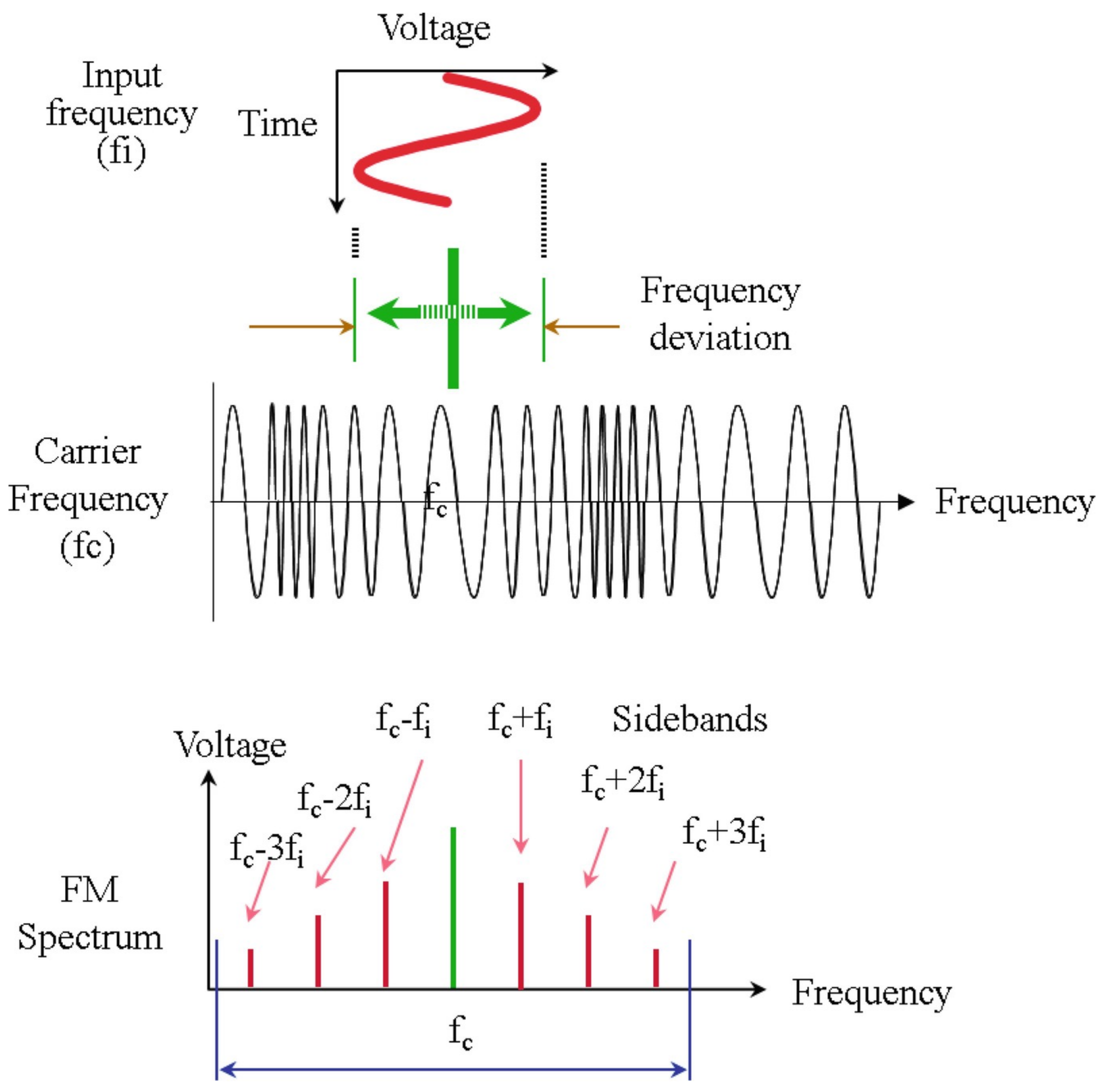
$$\cos(t\cos x) = J_0(t) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(t)\cos(2kx)$$

$$\sin(t\cos x) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(t)\cos\big((2k+1)x\big)$$

where J_i are Bessel functions.

FM Spectrum

- Those disgusting Bessel functions lead to what is technically an infinite number of sidebands in both directions
- The amount of modulation dictates the size/weight of those components.



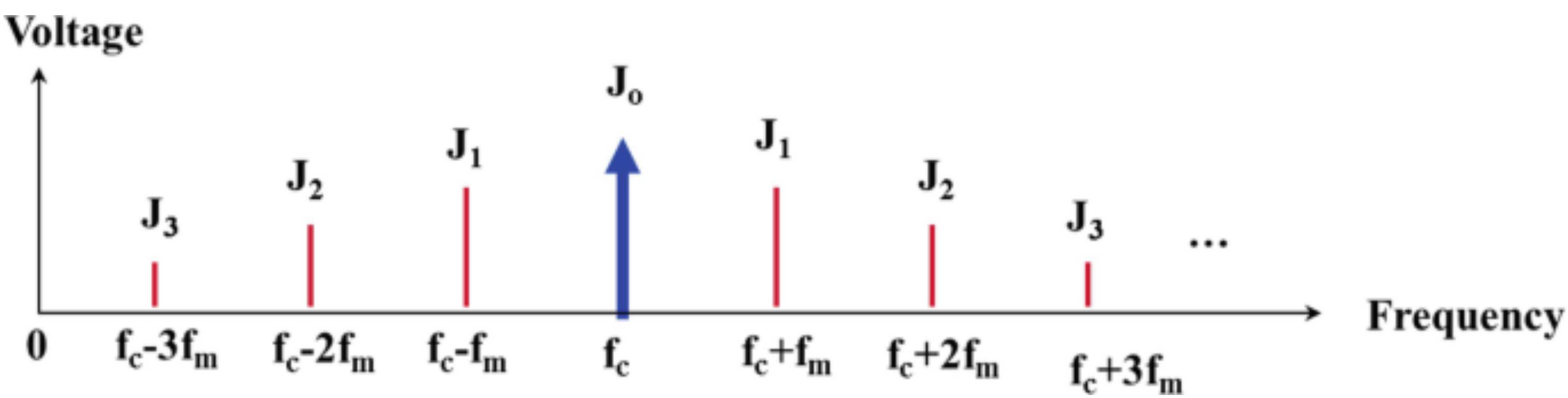
https://link.springer.com/chapter/10.1007/978-3-030-57484-0_8

$$v(t) = A\cos\big((\omega_c + \beta\,\omega_c \cos(\omega_m t))t\big)$$

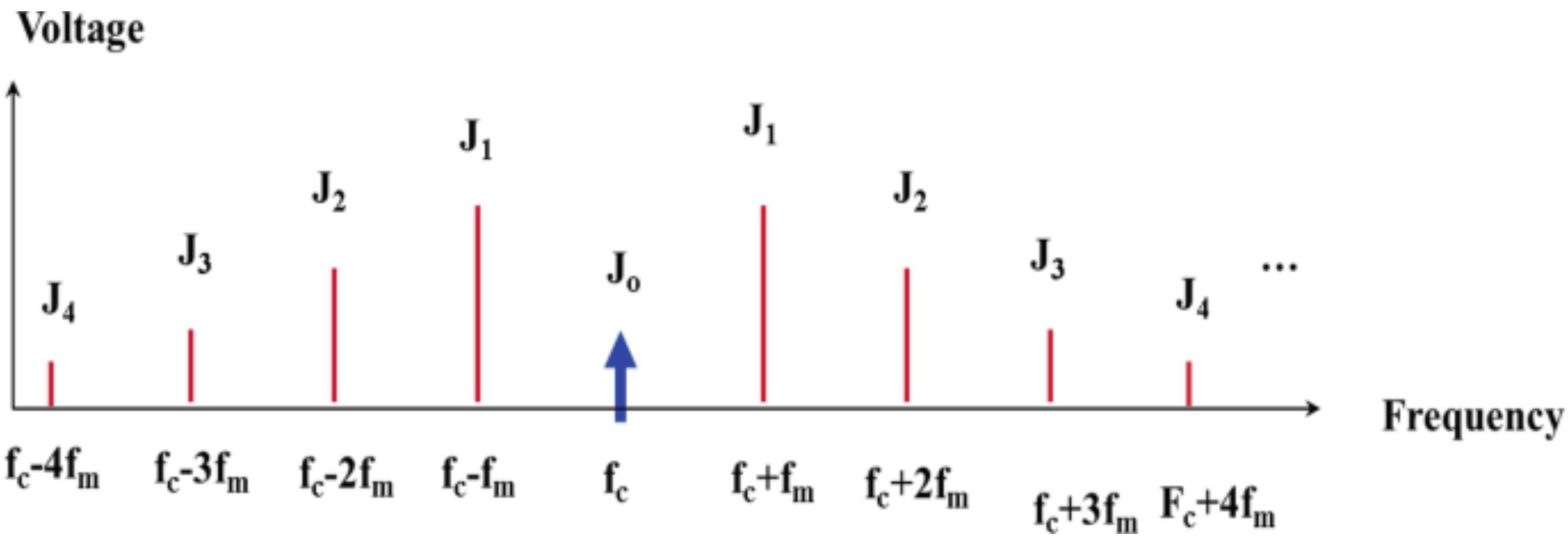
Carson's Rule

- β is the ratio ${}^{\Delta\omega}/_{\omega_c}$
- ~98% of the power of a FM signal is located within the bandwidth $2\omega_m(1+\beta)$

Low Frequency deviation…



Higher Frequency Deviation



https://link.springer.com/chapter/10.1007/978-3-030-57484-0_8

Carson’s Rule

- β is the ratio ${}^{\Delta\omega}/_{\omega_c}$

- ~98% of the power of a FM signal is located within the bandwidth $2\omega_m(1+\beta)$

- The more you modulate the larger the spectrum you need to harvest on capture

FM recovery/demodulation

- Unlike in AM where no matter what percentage of your carrier you modulate you have the same bandwidth as your signal (2X)

- In FM, the more you modulate, the more it pushes recoverable energy out to further harmonics from the Bessel functions, meaning you need more bandwidth.

- So always a battle.

FM is used…

- In a decent number of places…it is very hard (though not impossible) to mess with the frequency of a wave, at least when compared to amplitude modulation, so FM is much better in regards to noise than AM

- Circuits can just automatically control

If you needed to convey information on this wave what could you do?

$$v(t) = A\cos(2\pi f t + \phi)$$

- You could:
  - Vary the phase (Phase modulation)

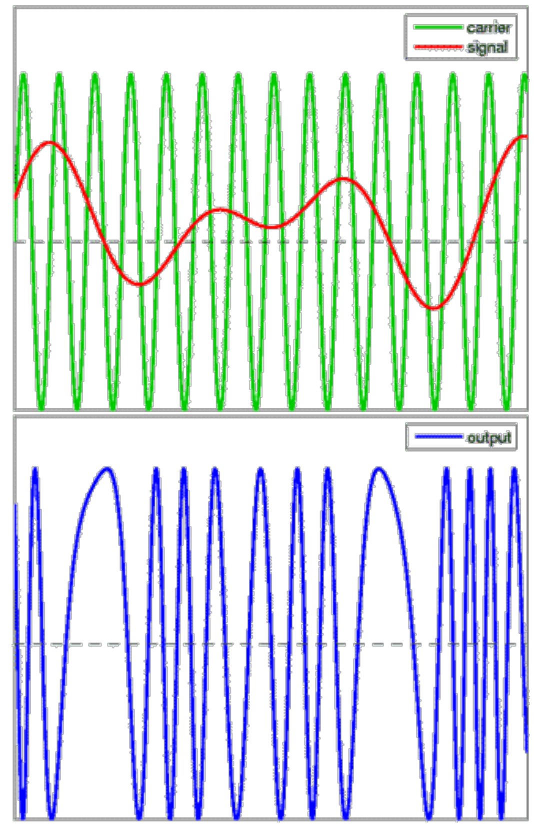
$$v(t) = A\cos(\omega t + \phi(t))$$

$$v(t) = A\cos(\omega_c t + \cos(\omega_m t))$$

Phase Modulation

- Varying the phase over time to convey your signal can be done

- But in a purely analog setting it is quite rare



https://commons.wikimedia.org/wiki/File:Phase_Modulation.png

Reason for that…

- Varying the phase of a signal over time starts to get tangled with the frequency of the signal

$$v(t) = A\cos(2\pi f t + \phi(t))$$

- Frequency really is just time-varying phase after all…

FM Modulation Looks Like...

$$v(t) = A\cos\big((\omega_c + \cos(\omega_m t))t\big)$$

Angle sum and difference identities [edit]

See also: Proofs of trigonometric identities § Angle sum identities, and Small-angle ap

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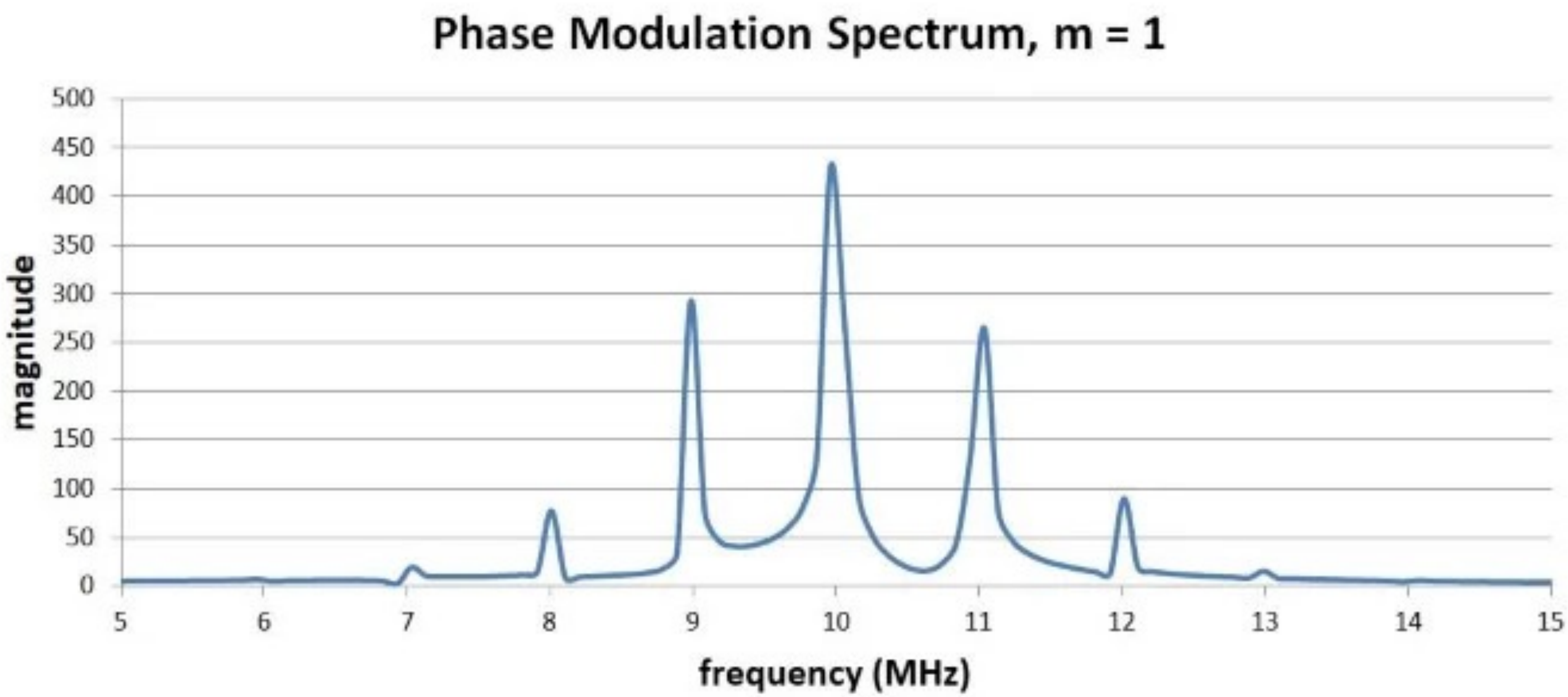
where J_i are Bessel functions.

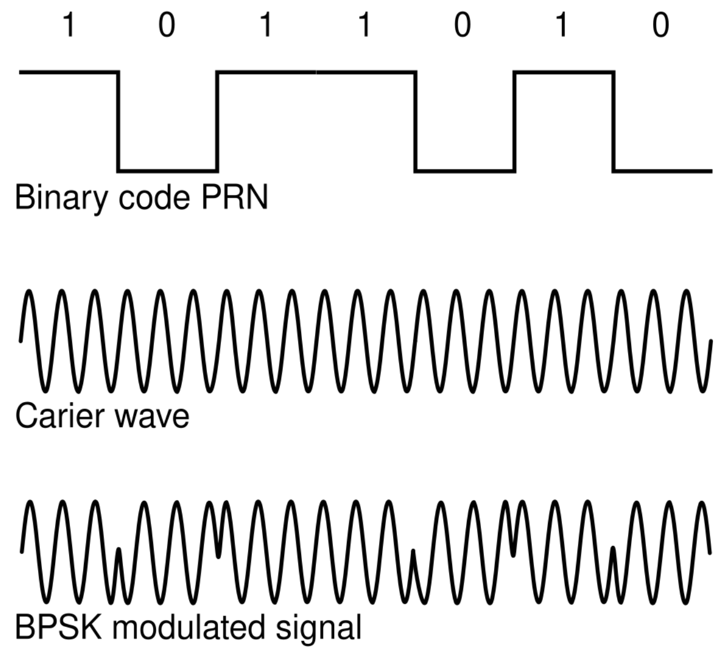
Since there’s still Bessel Functions

- A lot of the similar rules of frequency modulation can still apply to phase modulation…



Phase Modulation

- You do see phase modulation in lots of digital settings however. Distinct changes in the phase of a signal are easier to detect and less ambiguous than continuous “analog” phase modulation



https://commons.wikimedia.org/wiki/File:Phase_modulation_BPSK_GPS.svg

What do modern systems use?

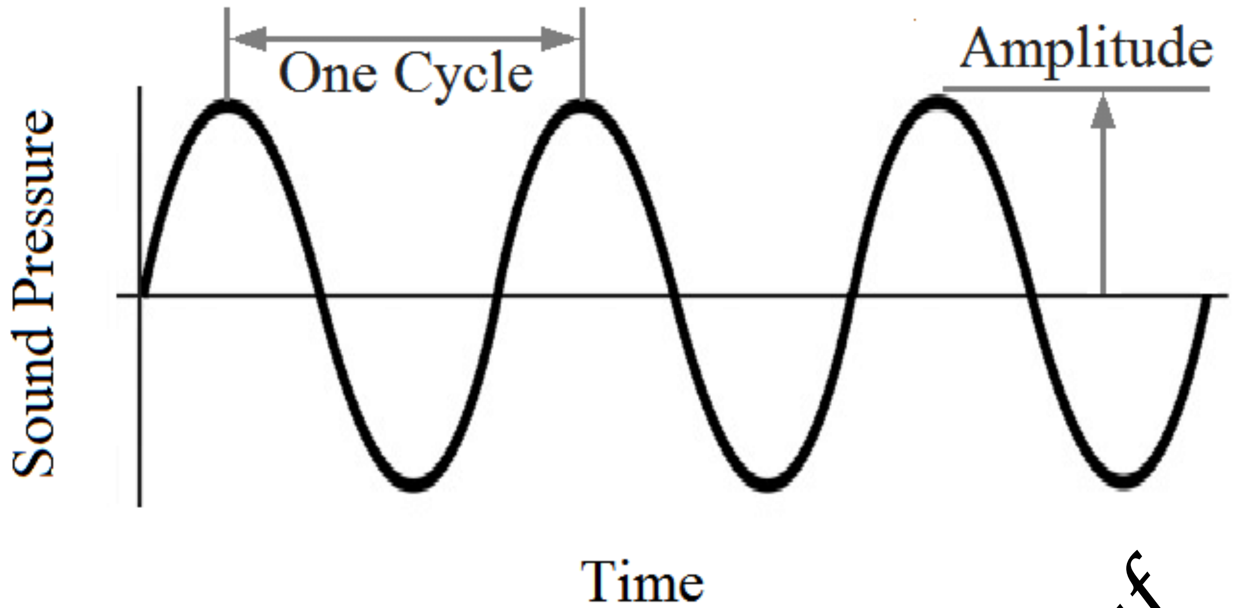
- Really depends
- Not a ton of regular pure-AM as we’ve seen it anymore
- FM still common in some applications
- PM is even more common and widely
- And as we’ll see some other combinations of the variants above is where many modern forms use, in particular even though AM on its own sucks, ***AM*** used a certain way with ***PM*** is how a lot of modern digitial data is transferred around.

So now…

- Let’s rethink our sine waves a bit.

- Because the way we’re viewing them so far has been limiting.

Sine wave looks like this



$$v(t) = A\sin(\omega t + \phi)$$

$$\omega = 2\pi f$$

Situation $v(t) = A(t)\cos(\omega_s t + \phi(t))$

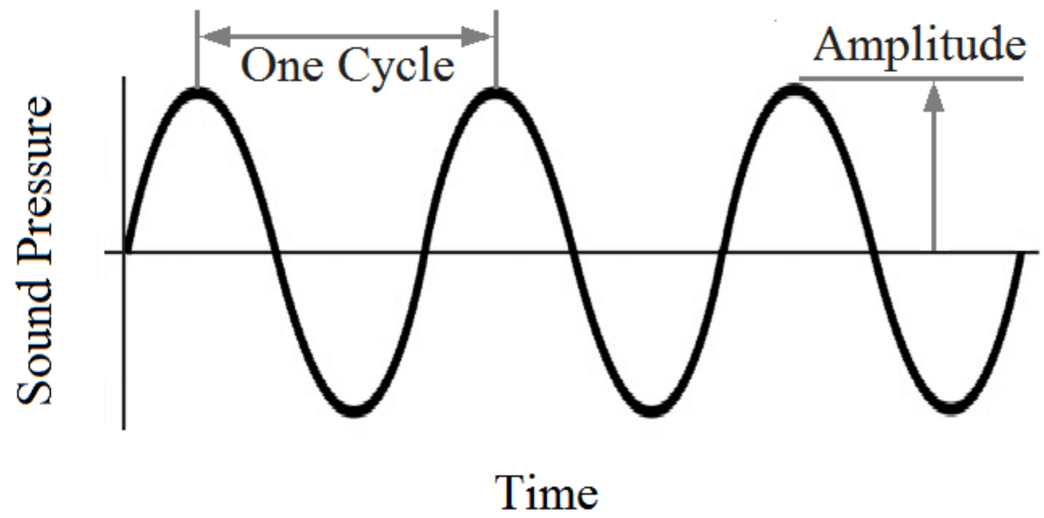
- Let's say you want to measure an incoming sine wave with some fixed known frequency ω_s and you want to determine what the amplitude and phase are

- How would you do that?

Problems $v(t) = A(t)\cos(\omega_s t + \phi(t))$

- One measurement is not enough to determine both A and ϕ

Lots of ambiguity in that lone measurement



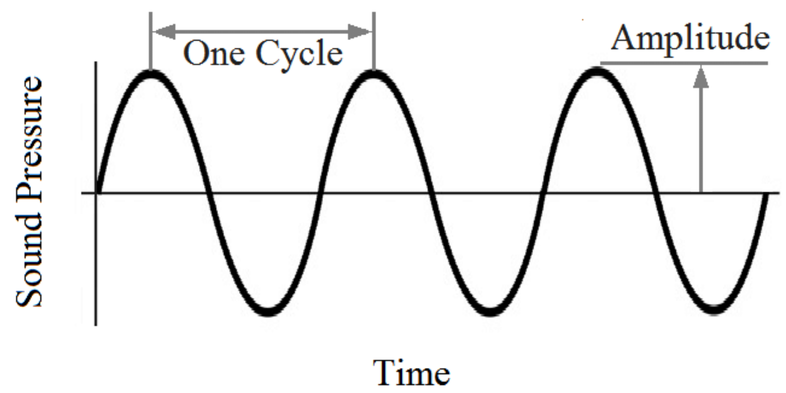
Solutions?

$$v(t) = A\cos(\omega_s t + \phi)$$

- You would need at least two measurements of the signal amplitude to be able to solve a system of equations yielding A and ϕ

$$v(t_1) = A\cos(\omega_s t_1 + \phi)$$

$$v(t_2) = A\cos(\omega_s t_2 + \phi)$$



Problems with that Solution?

$$v(t_1) = A(t)\cos(\omega_s t_1 + \phi(t))$$

$$v(t_2) = A(t)\cos(\omega_s t_2 + \phi(t))$$

uhhhhhh

- You're assuming the amplitude and phase stayed the same between those two points….that's potentially problematic if those two things are themselves varied over time to convey information.

The Real Problem…

- We’re unfortunately not thinking about oscillations in their true manner.

- We’re only thinking about them in one dimension when it can be more productive to think of them as two-dimensional entities.

Complex Numbers

- It is all about this thing: $e^{j\alpha}$

- Euler's Formula is this:

$$e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$$

- A special case is Euler's Identity (when $\alpha = \pi$)

$$e^{j\pi} = -1$$

Euler's Identity and Formula Are Not Up for Debate

Euler's formula is ubiquitous in mathematics, physics, chemistry, and engineering. The physicist Richard Feynman called the equation "our jewel" and "the most remarkable formula in mathematics".[2]

Mathematical beauty [edit]

Euler's identity is often cited as an example of deep mathematical beauty.[5] Three of the basic arithmetic operations occur exactly once each: addition, multiplication, and exponentiation. The identity also links five fundamental mathematical constants:[6]

- The number 0, the additive identity
- The number 1, the multiplicative identity
- The number π (π = 3.14159...), the fundamental circle constant
- The number e (e = 2.71828...), also known as Euler's number, which occurs widely in mathematical analysis
- The number i, the imaginary unit such that $i^2 = -1$

The equation is often given in the form of an expression set equal to zero, which is common practice in several areas of mathematics.

Stanford University mathematics professor Keith Devlin has said, "like a Shakespearean sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler's equation reaches down into the very depths of existence".[7] Paul Nahin, a professor emeritus at the University of New Hampshire who wrote a book dedicated to Euler's formula and its applications in Fourier analysis, said Euler's identity is "of exquisite beauty".[8]

Mathematics writer Constance Reid has said that Euler's identity is "the most famous formula in all mathematics".[9] Benjamin Peirce, a 19th-century American philosopher, mathematician, and professor at Harvard University, after proving Euler's identity during a lecture, said that it "is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth".[10]

A 1990 poll of readers by *The Mathematical Intelligencer* named Euler's identity the "most beautiful theorem in mathematics".[11] In a 2004 poll of readers by *Physics World*, Euler's identity tied with Maxwell's equations (of electromagnetism) as the "greatest equation ever".[12]

At least three books in popular mathematics have been published about Euler's identity:

- *Dr. Euler's Fabulous Formula: Cures Many Mathematical Ills*, by Paul Nahin (2011)[13]
- *A Most Elegant Equation: Euler's formula and the beauty of mathematics*, by David Stipp (2017)[14]
- *Euler's Pioneering Equation: The most beautiful theorem in mathematics*, by Robin Wilson (2018)[15]

Euler's formula · half-lives (exponential growth and decay)

Defining e
proof that e is irrational · representations of e · Lindemann–Weierstrass theorem

People
John Napier · Leonhard Euler

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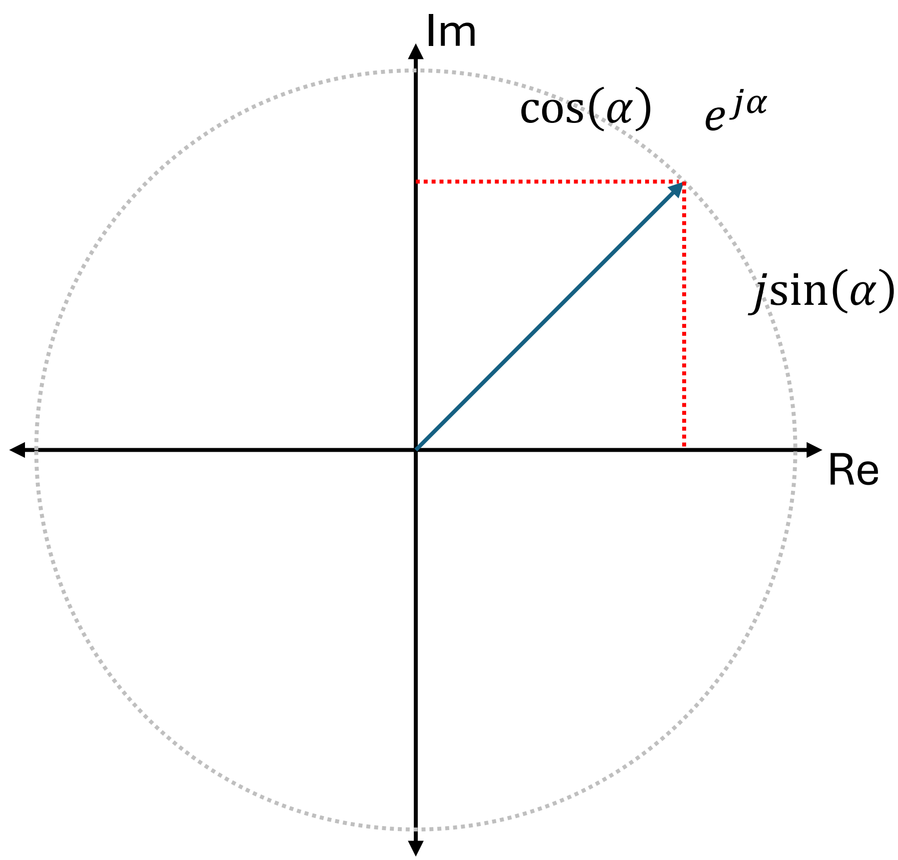
From Wikipedias

$e^{j\alpha}$ lives in the Complex Plane

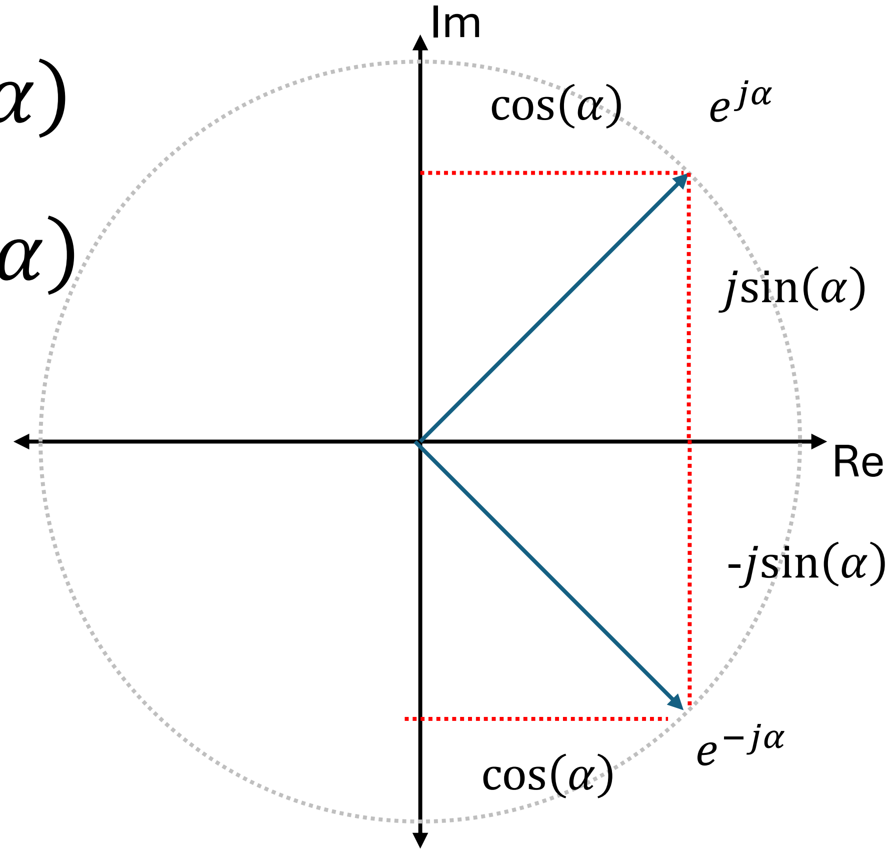
- You have a Real and Imaginary component to $e^{j\alpha}$ and it varies with α.



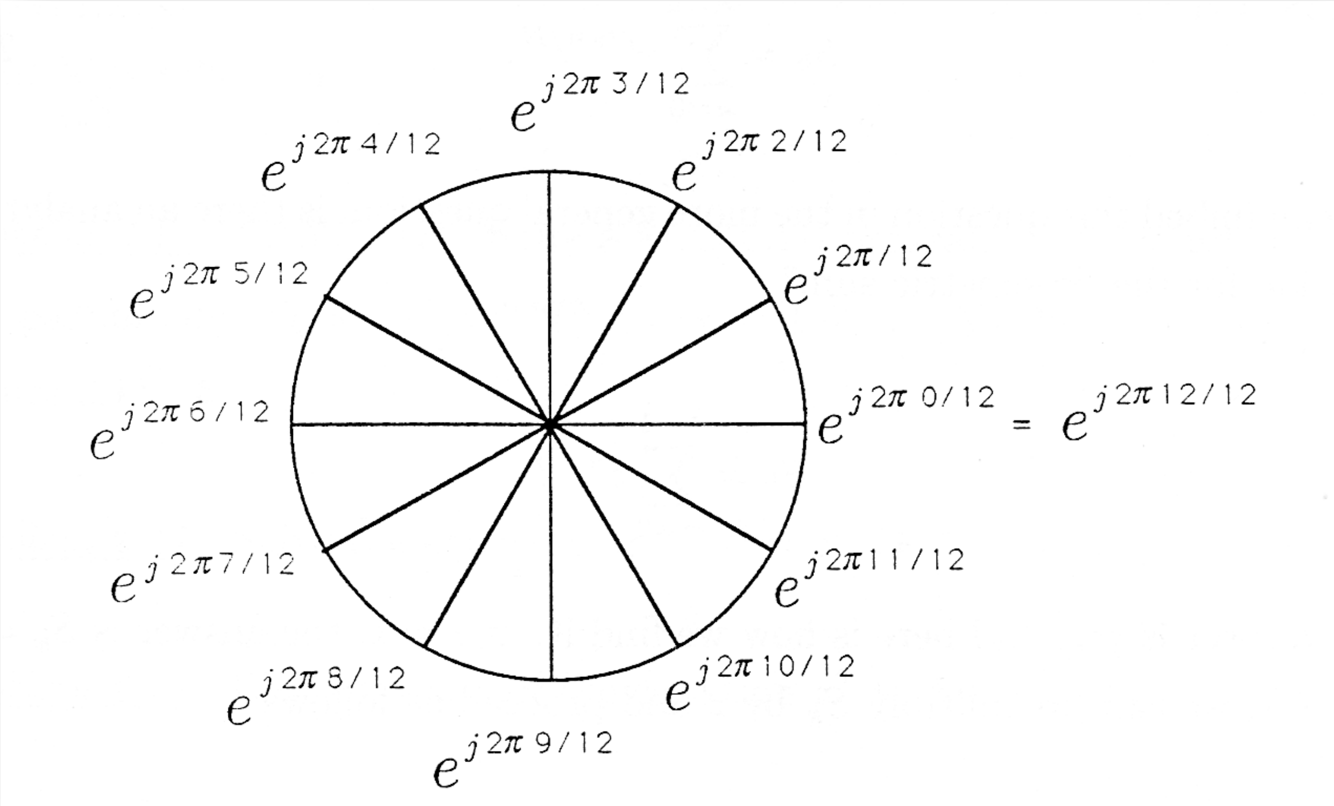
$e^{j\alpha}$ lives in the Complex Plane

$$e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$$

$$e^{-j\alpha} = \cos(\alpha) - j\sin(\alpha)$$

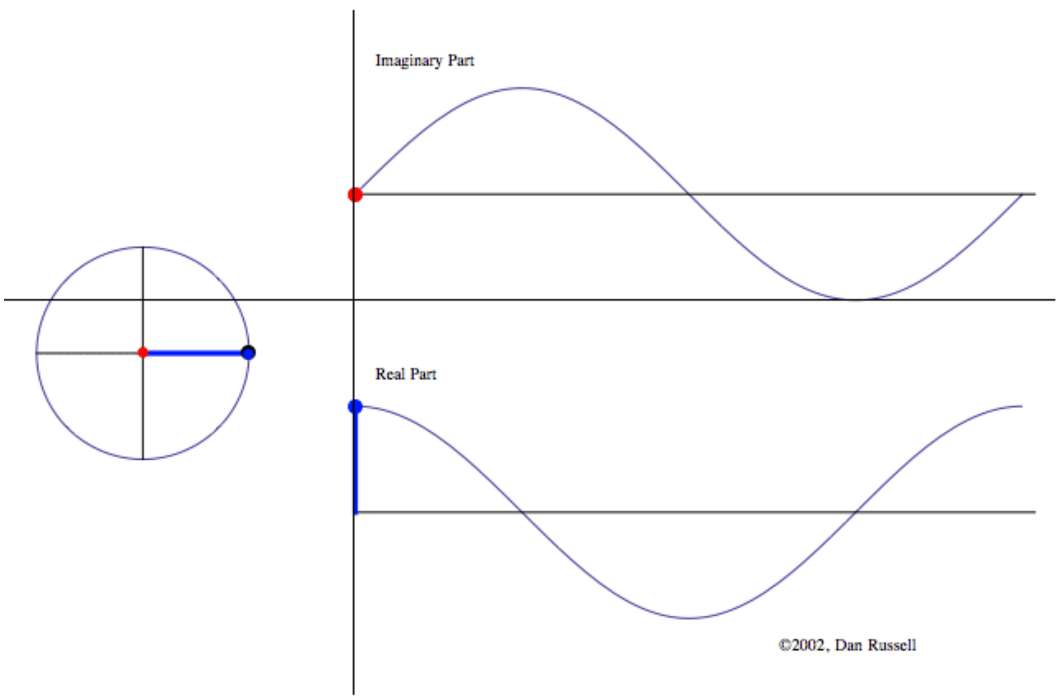


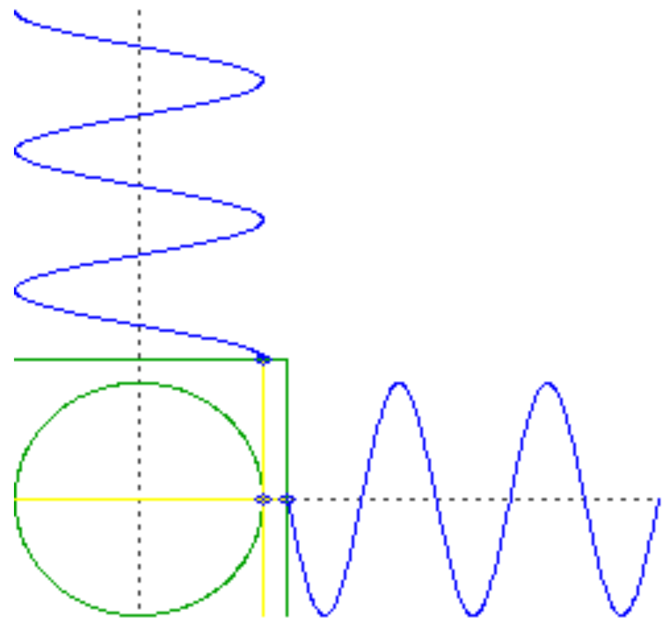
And of course as you vary α, the location of $e^{j\alpha}$ moves around the complex plane



https://math.stackexchange.com/questions/72113/has-anyone-talked-themselves-into-understanding-eulers-identity-a-bit

$$e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$$





https://www.acs.psu.edu/drussell/Demos/complex/complex.html

Can also repackage that

$$e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$$

$$e^{-j\alpha} = \cos(\alpha) - j\sin(\alpha)$$

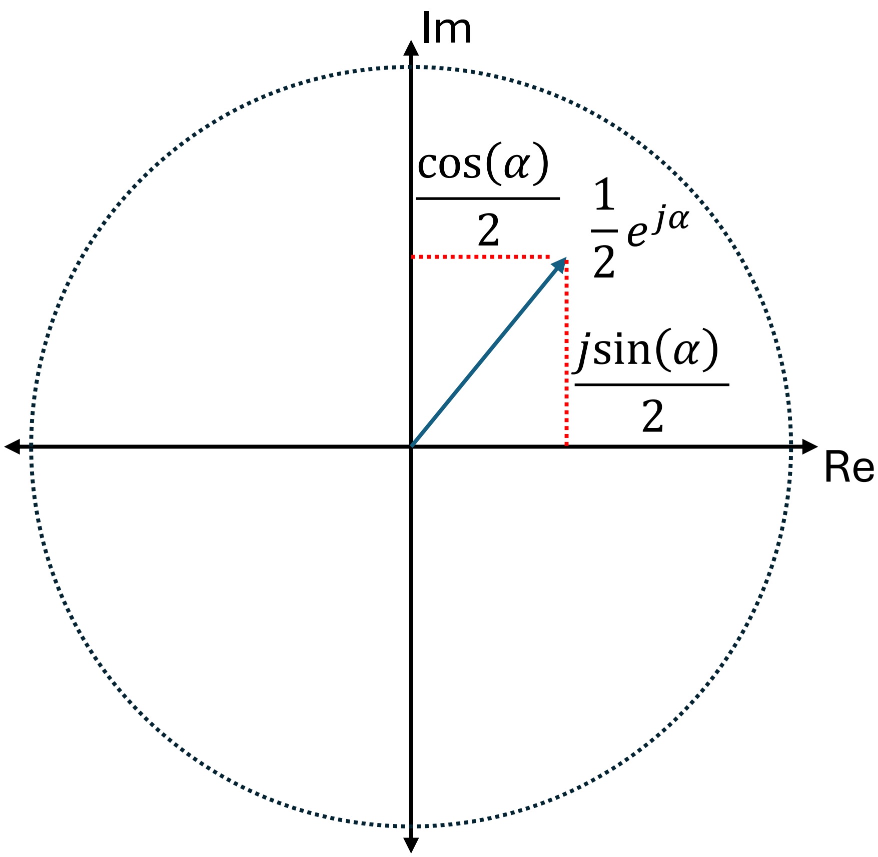
- Use these…to get these:

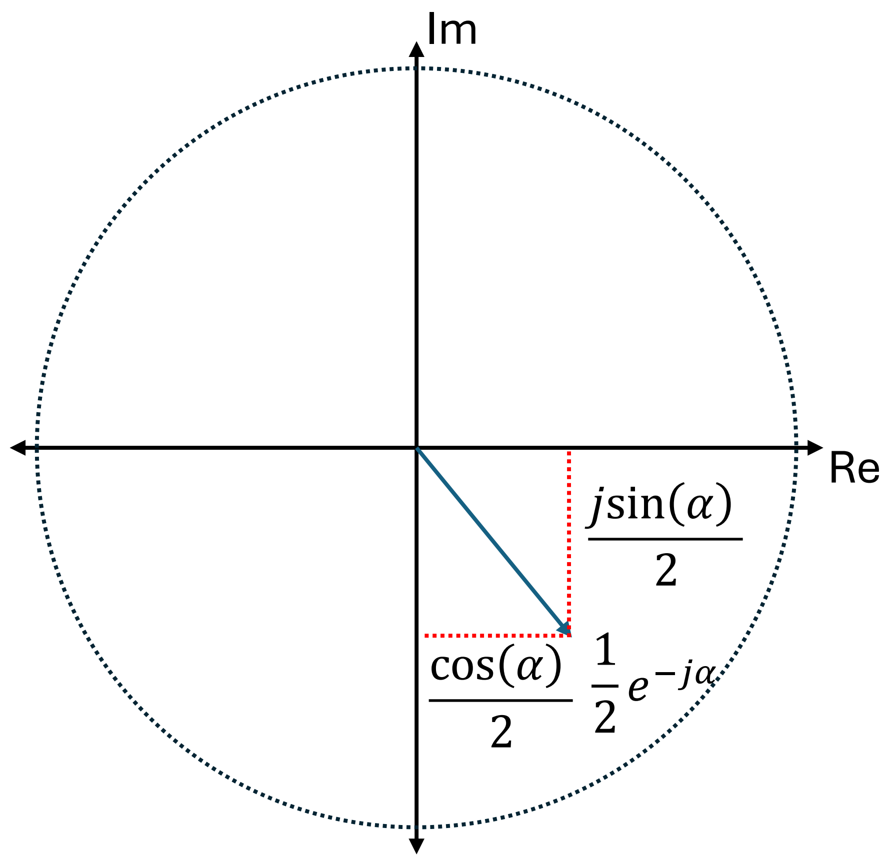
$$\cos(\alpha) = \mathrm{Re}\{e^{j\alpha}\} = \frac{e^{j\alpha}}{2} + \frac{e^{-j\alpha}}{2}$$

$$\sin(\alpha) = \mathrm{Im}\{e^{j\alpha}\} = \frac{-je^{j\alpha}}{2} + \frac{je^{-j\alpha}}{2}$$

Each Term of what a cosine is:

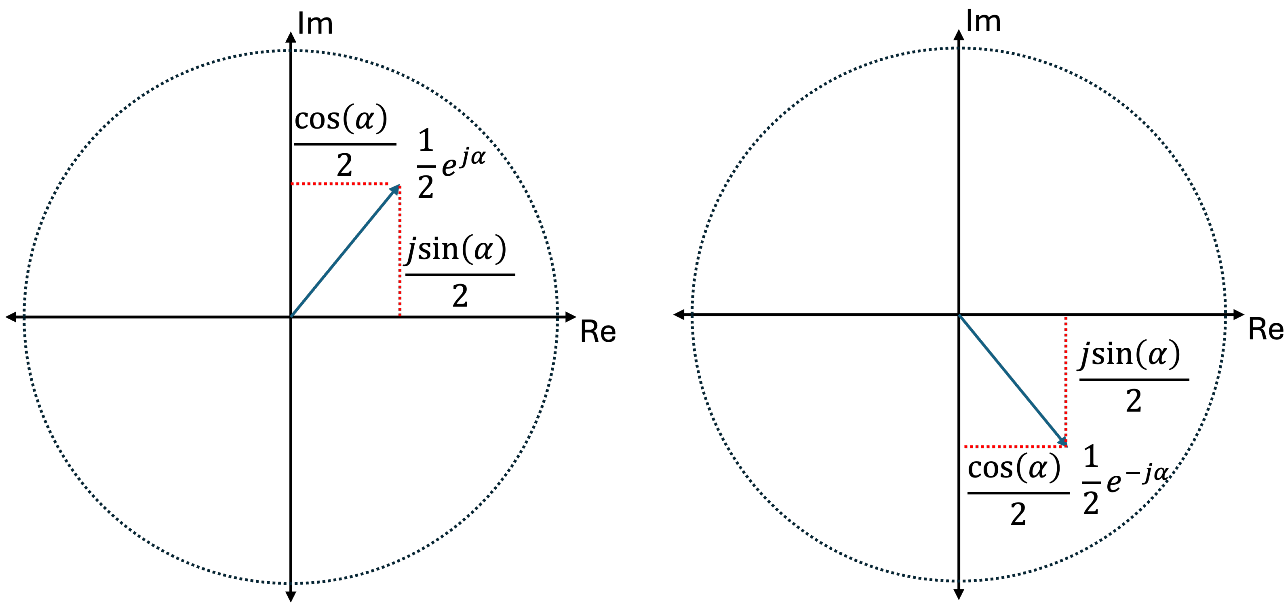
$$\cos(\alpha) = \mathrm{Re}\{e^{j\alpha}\} = \frac{e^{j\alpha}}{2} + \frac{e^{-j\alpha}}{2}$$





Each Term of what a cosine is:

$$\cos(\alpha) = \mathrm{Re}\{e^{j\alpha}\} = \frac{e^{j\alpha}}{2} + \frac{e^{-j\alpha}}{2}$$



The imaginary parts are always canceling out, leaving just the real parts.

It isn't a hard leap to replace α with our ωt from before

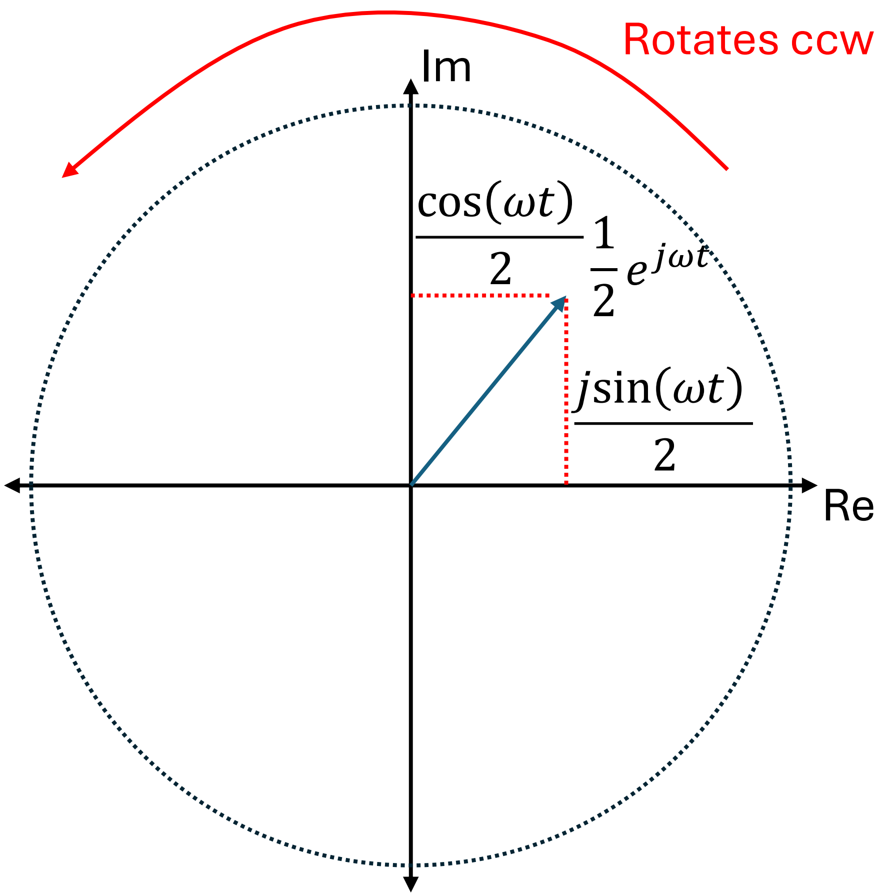
- Use these…to get these:

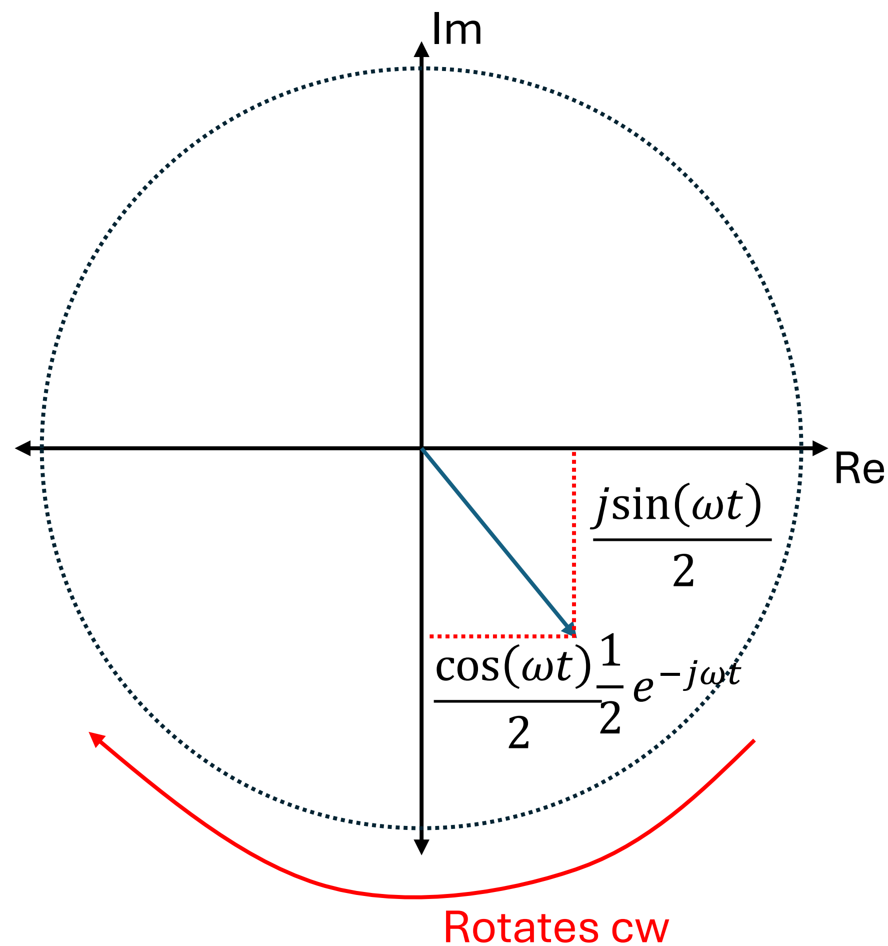
$$\cos(\omega t) = \frac{e^{j\omega t}}{2} + \frac{e^{-j\omega t}}{2}$$

$$\sin(\alpha) = \frac{-je^{j\omega t}}{2} + \frac{je^{-j\omega t}}{2}$$

Each Term of what a cosine is:

$$\cos(\omega t) = \text{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t}}{2} + \frac{e^{-j\omega t}}{2}$$





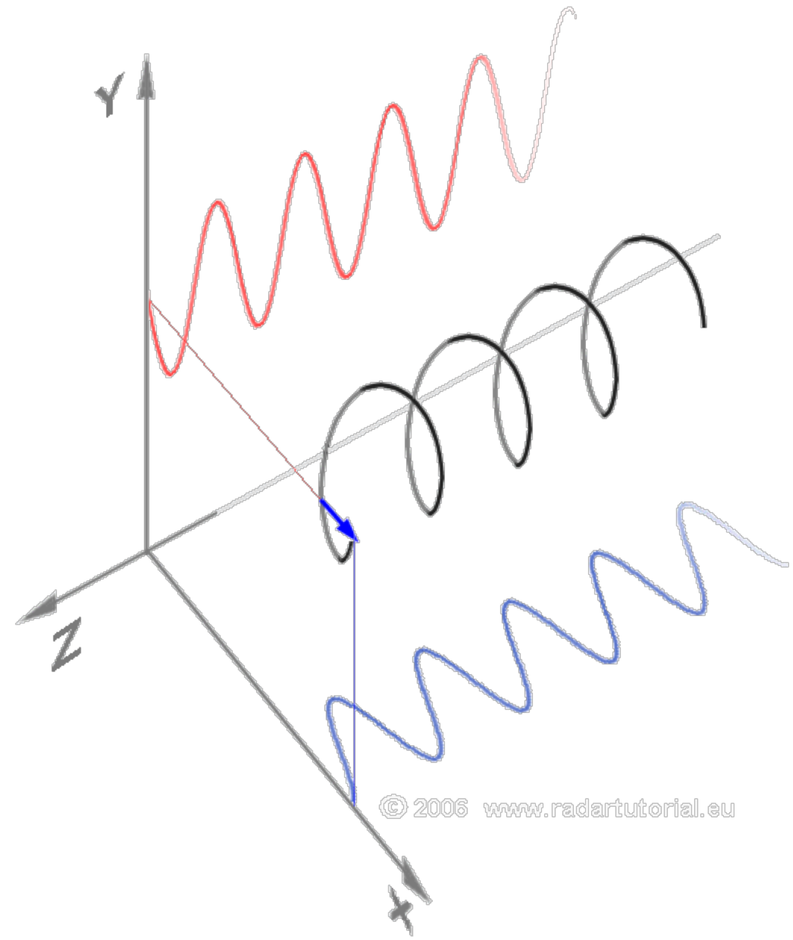
Call each of the components

- Sinusoids can therefore be thought of as sums of complex oscillating signals

$$\cos(\omega t) = \frac{e^{j\omega t}}{2} + \frac{e^{-j\omega t}}{2}$$

$$\sin(\alpha) = \frac{-je^{j\omega t}}{2} + \frac{je^{-j\omega t}}{2}$$

- Individual complex signals through time



Don’t have a good animation for superimposed pair, but individuals will show up in real and imaginary axis

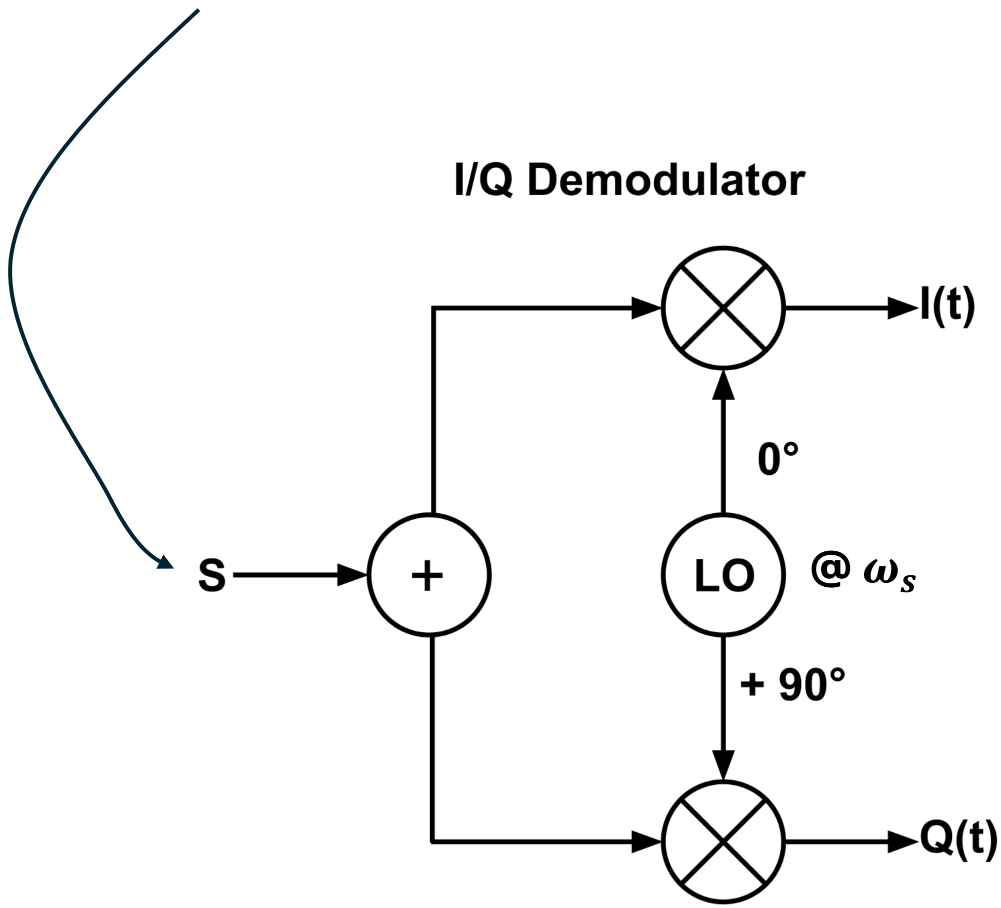
Return to situation

$$v(t) = A(t)\cos(\omega_s t + \phi(t))$$

- Let’s say you want to measure an incoming sine wave with some fixed known frequency ω_s and you want to determine what the amplitude and phase are

- How would you do that?

$$v(t) = A(t)\cos(\omega_s t + \phi(t))$$



https://en.wikipedia.org/wiki/In-phase_and_quadrature_components#/media/File:IQ_Mod_Demod_block_2.svg